ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2016, Vol. 37, No. 6, pp. 713–722. © Pleiades Publishing, Ltd., 2016.

Elementary Theories and Structural Properties of D-C.E. and N-C.E. Degrees

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Received April 30, 2016

Abstract—This paper is a survey on the upper semilattices of Turing and enumeration degrees of n-c.e. sets. Questions on the structural properties of these semilattices, and some model-theoretic properties are considered.

DOI: 10.1134/S199508021606010X

Keywords and phrases: *Erhsov's hierarchy, Turing degrees, enumeration degrees, elementary theories, structural properties.*

1. n-c.e. TURING DEGREES

The notion of a *computably enumerable (c.e.) set*, i.e. a set of integers whose members can be effectively listed, is a fundamental one. Another way of approaching this definition is via an approximating function $\{A_s\}_{s\in\omega}$ to the set A in the following sense: we begin by guessing $x \notin A$ at stage 0 (i.e. $A_0(x) = 0$); when later x enters A at a stage $s+1$, we change our approximation from $A_s(x) = 0$ to $A_{s+1}(x) = 1$. Note that this approximation (for fixed) x may change at most once as s increases, namely when x enters A. An obvious variation of this definition is to allow more than one change: a set A is *2-c.e.* (or *d-c.e.*) if for each x, $A_s(x)$ changes at most twice as s increases. This is equivalent to requiring the set A to be the difference of two c.e. sets $B_1 - B_2$. Similarly, one can define n-c.e. sets by allowing n changes for each x. The last is equivalent to an existence of c.e. sets $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n$ such that

$$A = (B_1 - B_2) \cup \cdots \cup (B_{n-1} - B_n),$$

if n is even, and

$$A = (B_1 - B_2) \cup \cdots \cup (B_{n-2} - B_{n-1}) \cup B_n,$$

if n is odd.

A direct generalization of this reasoning leads to sets which are computably approximable in the following sense: for a set A there is a set of uniformly computable sequences $\{f(0,x), f(1,x), ...f(s,x), ...|x \in \omega\}$ consisting of 0 and 1 such that for any x the limit of the sequence $f(0,x), f(1,x), ...$ exists and is equal to the value of the characteristic function $A(x)$ of A. The well-known Shoenfield Lemma states that the class of such sets coincides with the class of all Δ^0_2-sets. Thus, for a set A, $A \leqslant_T \emptyset'$ if and only if there is a computable function $f(s,x)$ such that $A(x) = \lim_s f(s,x)$.

The notion of d-c.e. and n-c.e. sets goes back to Putnam [35] and Gold [26] and was first investigated and generalized by Ershov [18]–[20]. The arising hierarchy of sets is now known as *the Ershov difference hierarchy*. The position of a set A in this hierarchy is determined by the number of changes

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